

Stimulated transitions due to arbitrary, Fourier decomposed, perturbing Hamiltonians

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An external, time-dependent perturbing potential acting upon a quantum system can lead to transitions between its eigenstates which oscillate in time. In this paper we present an approximate analytic solution for the temporal evolution of the system in response to an arbitrary perturbing Hamiltonian that has been decomposed into its Fourier modes. We impose no restriction upon the number of eigenstates of the system nor upon the structure of the perturbing Hamiltonian, the number of Fourier modes, their amplitude or their frequencies. New terms appear in our results indicating new avenues by which the frequencies and amplitudes of the modes may affect the transition probabilities. We consider two applications of the theory to neutrino flavor transformation through matter. The first is the effect of two sinusoidal density fluctuations upon a three flavor neutrino where we find the phenomenon of induced transparency. The second is the case of a mono-energetic, neutrino self-interaction Hamiltonian with constant coupling. In both applications we demonstrate how the analytic solutions are able to match the amplitude and wavenumber of the numerical results to within a few percent.

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I. INTRODUCTION

Determining the response of a quantum mechanical system to a time-dependent perturbation is a frequent endeavor of both experiment and theory in a large number of subfields of physics. A number of phenomena have been found to occur in quantum optics such as induced transparency and stimulated Raman adiabatic passage, in electronic spin and nuclear magnetic resonance, and in ultracold atoms and molecules to name just a few. Many reviews of driven quantum systems can be found including [1]. From the theory side, several approaches to the calculation of the transition probability between the states of the system can be found in textbook literature with the various techniques having strengths and weaknesses depending upon the form of the perturbation. Even when we restrict our attention to harmonic perturbations, one may compute the transition probability (or transition rate) between states using Floquet theory [2], the Rotating Wave Approximation (RWA) [3], Fermi's Golden Rule and others. A comparison between these techniques and more for a two-level system can be found in Dion & Hirschfelder [4].

Perhaps one of the more unusual cases of a driven quantum mechanical system is for the flavor evolution in space/time of a neutrino as it propagates through matter with periodic fluctuations. Both Floquet theory and the Rotating Wave Approximation approaches have been used to calculate the effect of density fluctuations described by a single Fourier mode [5–10]. More recently still Patton, Kneller and McLaughlin [11] considered the case of neutrino flavor evolution through *aperiodic* matter fluctuations as one would find in a turbulent medium. They found the neutrino transition probability for a two flavor neutrino did not evolve chaotically as one might expect but rather exhibited quasi periodicity by approximately following $P_{12} \approx A \sin^2(Qr)$. The RWA, generalized to an arbitrary number of modes with no restric-

tion upon their amplitudes or wavenumbers, was able to predict the wavenumber Q and the amplitude A . An alternative analysis of the similar problem of a two-level atom interacting with a stochastic electromagnetic field is found in Cummings [17] and the evolution of neutrinos through random matter density fluctuations has been previously considered by many others [18–32].

But neutrinos propagating through matter fluctuations / irradiated dipolar molecules are special cases because the perturbing Hamiltonian for both problems have the same particular structure. The solutions found by Brown, Meath and Tran and Patton, Kneller and McLaughlin are restricted to only those quantum systems which can be reduced to the same form. But the good agreement found between the RWA predictions and the exact numerical solutions leads one to speculate whether a more general RWA method for perturbing Hamiltonians with arbitrary structure would open a larger class of problems to analysis.

We present the generalization of the RWA method for the case of an arbitrary, Fourier decomposed perturbing Hamiltonian in this paper. The theory is described in section §II in the most general terms possible without reference to any particular quantum system. We then go on to show how the theory can be used in practice in section §III by using it to analyze the effect of two sinusoidal density fluctuations upon a three flavor neutrino system, and a non-linear, ‘self-interaction’ neutrino evolution model often used in the literature. These test problems are sufficiently simple that they illustrate how the RWA is applied in practice while at the same time, being sufficiently novel and interesting that the results may be of use to those studying neutrino evolution in practice.

II. STIMULATED TRANSFORMATION

The general problem we wish to consider is the evolution in time of a arbitrary N level quantum system due to some time dependent perturbation. At some initial time t_1 we prepare the system in some arbitrary state - represented by a column vector - which we decompose into a sum of the N states of some basis (X). The system then evolves to a time t_2 and at which we decompose the state in terms of the N states of a possibly different basis (Y). The evolution is described by a matrix $S^{(YX)}(t_2, t_1)$ and the transition probabilities are the set of probabilities that the system in a given initial state x of (X) at t_1 is detected in the state y of (Y) at t_2 . These transition probabilities are denoted by $P_{yx}^{(YX)}(t_2, t_1)$ and are related to the elements of $S^{(YX)}$ by $P_{yx}^{(YX)} = |S_{yx}^{(YX)}|^2$. Since $S^{(YX)}$ must be unitary, one needs N^2 independent real parameters in order to describe the matrix $S^{(YX)}$ but note only $(N-1)^2$ of the elements of $P^{(YX)}$ are independent. Hereafter we shall work with the case where the bases (X) and (Y) are the same although there are certainly circumstances where knowing the evolution from one basis to a different basis is useful. Note also that throughout this paper we set $\hbar = c = 1$.

In some generic basis (Z) the evolution matrix can be found by solving the Schrodinger equation

$$i \frac{dS^{(ZZ)}}{dt} = H^{(Z)} S^{(ZZ)} \quad (1)$$

where $H^{(Z)}$ is the Hamiltonian in the basis (Z). The initial condition is $S^{(ZZ)}(t_1, t_1) = 1$. We make no assumption about the structure of $H^{(Z)}$ except that it be possible to separate the Hamiltonian into an unperturbed piece $\check{H}^{(Z)}(t)$ and a position dependent perturbation $\delta H^{(Z)}(t)$ i.e. $H^{(Z)}(t) = \check{H}^{(Z)}(t) + \delta H^{(Z)}(t)$.

If $\check{H}^{(Z)}(t)$ is not diagonal then we introduce an instantaneous unperturbed eigenbasis (u) by finding the unitary matrix $\check{U}(t)$ defined by $\check{H}^{(Z)} = \check{U} \check{K} \check{U}^\dagger$ where \check{K} is the diagonal matrix of the eigenvalues of \check{H} , that is $\check{K} = \text{diag}(\check{k}_1, \check{k}_2, \dots)$. The evolution matrix in the instantaneous unperturbed eigenbasis is related to the evolution $S^{(ZZ)}$ by $S^{(uu)}(t_2, t_1) = \check{U}^\dagger(t_2) S^{(ZZ)}(t_2, t_1) \check{U}(t_1)$

In this unperturbed eigenbasis

$$H^{(u)} = \check{K} - i \check{U}^\dagger \frac{d\check{U}}{dt} + \check{U}^\dagger \delta H^{(Z)} \check{U} \quad (2)$$

We now write the evolution matrix in the unperturbed eigenbasis as the product $S^{(uu)} = \check{S} A$ where \check{S} is defined to be the solution of

$$i \frac{d\check{S}}{dt} = \left[\check{K} - i \check{U}^\dagger \frac{d\check{U}}{dt} \right] \check{S}. \quad (3)$$

If we know the solution to the unperturbed problem, \check{S} , we can solve for the effect of the perturbation by finding

the solution to the differential equation for A :

$$i \frac{dA}{dt} = \check{S}^\dagger \check{U}^\dagger \delta H^{(Z)} \check{U} \check{S} A. \quad (4)$$

In general the term $\check{U}^\dagger \delta H^{(Z)} \check{U}$ which appears in this equation possesses both diagonal and off-diagonal elements. The diagonal elements are easily removed by writing the matrix A as $A = W B$ where $W = \exp(-i\Xi)$ and Ξ a diagonal matrix $\Xi = \text{diag}(\xi_1, \xi_2, \dots)$. Substitution into (4) gives a differential equation for B

$$i \frac{dB}{dt} = W^\dagger \left[\check{S}^\dagger \check{U}^\dagger \delta H^{(Z)} \check{U} \check{S} - \frac{d\Xi}{dt} \right] W B \equiv H^{(B)} B \quad (5)$$

and Ξ is chosen so that $d\Xi/dt$ removes the diagonal elements of $\check{S}^\dagger \check{U}^\dagger \delta H^{(Z)} \check{U} \check{S}$. Once Ξ has been found, determining transition probabilities is reduced to solving dB/dt .

A. Fourier Decomposed Perturbations

We now consider the specific case of a constant potential for $\check{H}^{(Z)}$. This form for \check{H} means \check{S} is a diagonal matrix $\check{S} = \exp(-i\check{K}t)$. The perturbation δH is taken to be a Fourier series of the form

$$\delta H^{(Z)} = \sum_a (C_a e^{i q_a t} + C_a^\dagger e^{-i q_a t}) \quad (6)$$

where C_a is an arbitrary complex matrix and q_a the frequency of the a 'th Fourier mode of the perturbation. We make no restriction on the number of Fourier modes, the frequencies q_a nor the size or structure of the matrices C_a . This generalization to arbitrary structure for the C_a 's is where we depart from previous analyses by Patton, Kneller & McLaughlin [11]. We also refer the reader to Brown, Meath & Tran [15] and Avetissian, Avchyan & Mkrtchian [16] who considered the related but simpler problem of the effect of two lasers of different colors, i.e. two Fourier modes, upon a two-level dipolar molecule.

Given this form for the perturbation, equation (5) indicates we need to consider the combination $\check{U}^\dagger C_a \check{U}$. If we write the diagonal elements of $\check{U}^\dagger C_a \check{U}$ as

$$\text{diag}(\check{U}^\dagger C_a \check{U}) = \frac{F_a}{2i} \exp(i \Phi_a) \quad (7)$$

where F_a is a diagonal matrix of amplitudes $F_a = \text{diag}(f_{a;1}, f_{a;2}, \dots)$ and Φ_a the diagonal matrix of phases $\Phi_a = \text{diag}(\phi_{a;1}, \phi_{a;2}, \dots)$, then the matrix Ξ is found $\Xi(t) = \sum_a \Xi_a(t)$ with

$$\Xi_a(t) = \frac{F_a}{q_a} [\cos \Phi_a - \cos(\Phi_a + q_a t)]. \quad (8)$$

We denote the diagonal elements of Ξ_a as $\xi_{a;1}, \xi_{a;2}, \dots$. Next we rewrite the off-diagonal elements of $\check{U}^\dagger C_a \check{U}$ as a matrix G_a i.e $G_a = \text{offdiag}(\check{U}^\dagger C_a \check{U})$. Putting together

the solution for Ξ and \check{S} and inserting the new matrix G_a , we find the Hamiltonian for B is

$$H^{(B)} = \exp(\imath \Xi) \exp(\imath \check{K} t) \left(\sum_a [G_a e^{\imath q_a t} + G_a^\dagger e^{-\imath q_a t}] \right) \times \exp(-\imath \check{K} t) \exp(-\imath \Xi) \quad (9)$$

Written explicitly the element ij of the Hamiltonian is

$$H_{ij}^{(B)} = e^{\imath(\delta \check{k}_{ij} t + \delta \xi_{ij})} \sum_a [G_{a;ij} e^{\imath q_a t} + G_{a;ji}^* e^{-\imath q_a t}] \quad (10)$$

where $\delta \check{k}_{ij} = \check{k}_i - \check{k}_j$ and $\delta \xi_{ij} = \xi_i - \xi_j$. If we define

$$x_{a;ij} = \frac{f_{a;i}}{q_a} \cos \phi_{a;i} - \frac{f_{a;j}}{q_a} \cos \phi_{a;j} \quad (11)$$

$$y_{a;ij} = \frac{f_{a;i}}{q_a} \sin \phi_{a;i} - \frac{f_{a;j}}{q_a} \sin \phi_{a;j} \quad (12)$$

$$z_{a;ij} = \sqrt{x_{a;ij}^2 + y_{a;ij}^2} \quad (13)$$

$$\tan \psi_{a;ij} = y_{a;ij} / x_{a;ij}. \quad (14)$$

then the term $\delta \xi_{ij}$ is equal to

$$\delta \xi_{ij} = \sum_a [x_{a;ij} - z_{a;ij} \cos(q_a t + \psi_{a;ij})] \quad (15)$$

The presence of $y_{a;ij}$ in these equations is a new feature of the more general perturbing Hamiltonian we are considering. We now make use of the Jacobi-Anger expansion for $e^{\imath \delta \xi_{ij}}$

$$e^{\imath \delta \xi_{ij}} = \prod_a \left\{ e^{\imath x_{a;ij}} \sum_{m_a=-\infty}^{\infty} (-\imath)^{m_a} J_{m_a}(z_{a;ij}) \exp[\imath m_a (q_a t + \psi_{a;ij})] \right\}. \quad (16)$$

If we substitute this expansion into the expression for the elements of $H^{(B)}$ and define $\kappa_{a;ij,m_a}$ and $\lambda_{a;ij,m_a}$ to be

$$\lambda_{a,m_a;ij} = (-\imath)^{m_a} e^{\imath x_{a;ij}} J_{m_a}(z_{a;ij}) e^{\imath m_a \psi_{a;ij}} \quad (17)$$

$$\kappa_{a,m_a;ij} = (-\imath)^{m_a} e^{\imath x_{a;ij}} [G_{a;ji}^* J_{m_a+1}(z_{a;ij}) e^{\imath(m_a+1)\psi_{a;ij}} - G_{a;ij} J_{m_a-1}(z_{a;ij}) e^{\imath(m_a-1)\psi_{a;ij}}] \quad (18)$$

then the we find the element ij of the Hamiltonian is given by

$$H_{ij}^{(B)} = \imath \sum_a \left\{ \sum_{m_a} \kappa_{a,m_a;ij} e^{\imath(m_a q_a + \delta \check{k}_{ij})t} \prod_{b \neq a} \left[\sum_{m_b} \lambda_{b,m_b;ij} e^{\imath m_b q_b t} \right] \right\} \quad (19)$$

B. Rotating Wave Approximation

Even though we started with a very general perturbing Hamiltonian, we have found a form for $H^{(B)}$ which has the same structure as that found by Patton, Kneller & McLaughlin. From here on, we follow the same procedure to solve for the matrix B . First we adopt the Rotating Wave Approximation. The RWA amounts to selecting a particular value for the integers m_a and m_b in Eq.(19) and dropping all others. We do not specify a procedure for selecting those integers though algorithms exist. We expect there is not one procedure that can be adopted universally for all situations. There are some restrictions to be placed on the selection of the integers. In order that the resulting Hamiltonian be solvable we

cannot make choices for m_a and m_b for every element ij independently. Only $N - 1$ elements are to be regarded as independent and a suitable set could be either those on the sub/superdiagonal or the off-diagonal elements in a particular row or column. The values of m_a and m_b we select will be different for each independent element. If we call these chosen integers for element ij by $n_{a;ij}$ and define

$$\kappa_{ij} = \sum_a \kappa_{ij,n_{a;ij}} \prod_{b \neq a} \lambda_{ij,n_{b;ij}} \quad (20)$$

then $H_{ij}^{(B)}$ is simplified to

$$H_{ij}^{(B)} = -\imath \kappa_{ij} \exp \left[\imath \left(\sum_a n_{a;ij} q_a + \delta \check{k}_{ij} \right) t \right], \quad (21)$$

or, in full matrix, form

$$H^{(B)} = \begin{pmatrix} 0 & -\imath\kappa_{12} e^{\imath\left[\check{k}_{12} + \sum_a n_{a;12} q_a\right]t} & -\imath\kappa_{13} e^{\imath\left[\check{k}_{13} + \sum_a n_{a;13} q_a\right]t} & \dots \\ \imath\kappa_{12}^* e^{-\imath\left[\check{k}_{12} + \sum_a n_{a;12} q_a\right]t} & 0 & \imath\kappa_{12} e^{-\imath\left[\check{k}_{23} + \sum_a n_{a;23} q_a\right]t} & \dots \\ \imath\kappa_{23}^* e^{-\imath\left[\check{k}_{13} + \sum_a n_{a;13} q_a\right]t} & \imath\kappa_{23}^* e^{-\imath\left[\check{k}_{23} + \sum_a n_{a;23} q_a\right]t} & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (22)$$

Again, only $n_{a;12}$, $n_{a;23}$, $n_{a;34}$ etc. are independent: in all other cases the integer $n_{a;ij} = n_{a;il} + n_{a;\ell j}$. As shown by Patton, Kneller & McLaughlin, with this simplified Hamiltonian, Eq.(5), can be solved for the evolution matrix B and we reproduce their solution here for completeness. Since both $n_{a;ij} = n_{a;il} + n_{a;\ell j}$ and $\check{\delta}k_{ij} = \check{\delta}k_{il} + \check{\delta}k_{\ell j}$, we can factorize $H^{(B)}(t)$ into the form $H^{(B)}(t) = \Upsilon(t) M \Upsilon^\dagger(t)$ where the matrix M is a constant, i.e. it contains the couplings κ_{ij} only. The matrix Υ is of the form $\Upsilon(t) = \exp(\imath \Lambda t)$, where Λ is also a con-

stant matrix that depends only on $\check{\delta}k_{ij}$, the integer sets $\{n_{a;ij}\}$ and the frequencies q_a . Explicitly we can write

$$M = \begin{pmatrix} 0 & -\imath\kappa_{12} & -\imath\kappa_{13} & \dots \\ \imath\kappa_{12}^* & 0 & -\imath\kappa_{23} & \dots \\ \imath\kappa_{13}^* & \imath\kappa_{23}^* & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (23)$$

and one possible choice for the matrix Λ is

$$\Lambda = \begin{pmatrix} \check{k}_1 + \sum_a n_{a;1} q_a & 0 & 0 & \dots \\ 0 & \check{k}_2 + \sum_a n_{a;2} q_a & 0 & \dots \\ 0 & 0 & \check{k}_3 + \sum_a n_{a;3} q_a & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (24)$$

where $n_{a;i}$ are integers chosen so that $n_{a;i} - n_{a;j} = n_{a;ij}$. Using this factorization of $H^{(B)}(t)$ we find equation (5) can be rewritten as

$$\imath\Upsilon^\dagger \frac{dB}{dt} = M \Upsilon^\dagger B \quad (25)$$

Instead of solving for B we solve for the combination $\Omega = \Upsilon^\dagger B$. The differential equation for Ω is found to be

$$\imath \frac{d\Omega}{dt} = (M + \Lambda) \Omega = H^{(\Omega)} \Omega. \quad (26)$$

Since the matrix both M and Λ are constant matrices, the matrix $H^{(\Omega)}$ is also independent of t meaning Ω has the formal solution $\Omega(t) = \exp(-\imath H^{(\Omega)} t) \Omega(0)$. The solution for B is thus

$$B(t) = \Upsilon(t) \exp(-\imath H^{(\Omega)} t) \Upsilon^\dagger(0) B(0). \quad (27)$$

Now that we have the solution for B , the full evolution matrix in the basis (u) is $S = \check{S} W B$ but given that both \check{S} and W are diagonal matrices, the transition probability between the unperturbed eigenstates is simply the square magnitude of the off-diagonal element of B . Note that we have made no restrictions on the number of Fourier modes, the frequencies nor the size and structure of the amplitude matrices so this procedure for obtaining the RWA solution applies to both periodic *and* aperiodic Hamiltonians.

For the particular case of two flavors we can write out the solution succinctly (after dropping the 12 subscripts on $n_{a;12}$ and κ_{12}) by introducing the detuning frequency p via $2p = \check{\delta}k_{12} + \sum_a n_a q_a$ and the Rabi flopping frequency Q by $Q^2 = p^2 + \kappa^2$. Using these quantities, $B(t)$ for two flavors is found to be

$$B = \begin{pmatrix} e^{ipt} \left[\cos(Qt) - i \frac{p}{Q} \sin(Qt) \right] & -e^{ipt} \frac{\kappa}{Q} \sin(Qt) \\ e^{-ipt} \frac{\kappa^*}{Q} \sin(Qt) & e^{-ipt} \left[\cos(Qt) + i \frac{p}{Q} \sin(Qt) \right] \end{pmatrix} \quad (28)$$

and so we see the transition probability between the matter states 1 and 2 is again

$$P_{12} = |B_{12}|^2 = \frac{\kappa^2}{Q^2} \sin^2(Qt). \quad (29)$$

Remarkably a transition probability that varies as $\sin^2(Qt)$ is a generic prediction of *all* RWA solutions of a two-level quantum system when the perturbing Hamiltonian can be expressed as a Fourier series even if that series is aperiodic.

C. Degenerate RWA

If the Fourier modes are such that the ratio between any pair of frequencies, q_a and q_b , is a rational fraction then there are multiple sets of the integers $\{n\}$ which *all* have the same detuning frequency. One can account for these degenerate RWAs by simply adding them together so that κ_{ij} becomes

$$\kappa_{ij} = \sum_{\{n\}} \left[\sum_a \kappa_{ij, n_a; ij} \prod_{b \neq a} \lambda_{ij, n_b; ij} \right]. \quad (30)$$

If all ratios between the frequencies are rational then accounting for the degeneracy in the RWA becomes an exercise in combinatorics. Even so, typically one finds the sum is dominated by one set - the one where the sum of the absolute values of the integers $\{n\}$ is smallest.

III. APPLICATIONS

With the theory complete we now move on to testing whether it gives predictions which agree with numerical

solutions. There are numerous toy problems of varying complexity that one can consider for this purpose from a variety of different fields. We choose two neutrino problems: the first is to show the phenomenon of induced transparency in three neutrino flavors, and the second the neutrino self-interaction problem. Note, as commonly found in the literature on neutrino flavor transformation, we switch the variable from time t to position along the neutrino trajectory r where $r = ct$ since the neutrino wavepacket is localized in space and typically the energy of neutrinos is much larger than their mass hence they move at a speed close to c .

A. Neutrino Induced Transparency

Our first test problem is the case of a three flavor neutrino of energy 5 MeV passing through matter which oscillates as a function of distance according to two sinusoidal fluctuations not necessarily in a rational ratio. The Hamiltonian, H , governing the neutrino flavor evolution through matter is the sum of a constant vacuum term H_V and a term coming from the effect of matter H_M [33, 34]. The vacuum Hamiltonian in the ‘flavor basis’ is

$$H_V^{(f)} = \frac{1}{2E} U_V \begin{pmatrix} m_1^2 - m_2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_3^2 - m_2^2 \end{pmatrix} U_V^\dagger \quad (31)$$

where U_V is the vacuum mixing matrix and m_i the three neutrino masses. We set the squared mass differences $m_1^2 - m_2^2 = -7.5 \times 10^{-5} \text{ eV}^2$ and $m_3^2 - m_2^2 = 2.32 \times 10^{-3} \text{ eV}^2$ which are compatible with the mass-squared differences as given by the Particle Data Group [35]. This matrix is parameterized by three mixing angles θ_{12} , θ_{13} and θ_{23} - we set all possible phases to zero [36] - and given by

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23} & c_{12}c_{23} - s_{12}s_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23} & -c_{12}s_{23} - s_{12}s_{13}c_{23} & c_{13}c_{23} \end{pmatrix} \quad (32)$$

where the notation is that $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. We take the angles to be $\theta_{12} = 34^\circ$, $\theta_{13} = 9^\circ$ and $\theta_{23} = 45^\circ$ [35]. The matter Hamiltonian is taken to be

a constant upon which are superposed two sinusoidal oscillations with wavenumbers q_1 and q_2 . The matter is taken to affect only the electron flavor type, not the other

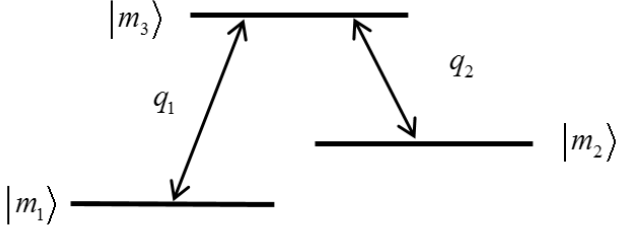


FIG. 1: The three eigenstates of the unperturbed neutrino induced transparency problem.

two flavors. The form of the Hamiltonian in the flavor basis with the first row/column indicating the electron flavor is thus

$$H_M^{(f)}(r) = V_* [1 + A_1 \cos(q_1 r) + A_2 \cos(q_2 r)] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (33)$$

with V_* the potential from the constant background, A_1 and A_2 the amplitudes of the fluctuations. In what follows we set this parameter to $V_* = 6 \times 10^{-25}$ erg.

The vacuum Hamiltonian and the constant potential V_* form the ‘unperturbed’ Hamiltonian \check{H} . In the flavor basis \check{H} is not diagonal. We can diagonalize \check{H} by first finding its eigenvalues, denoted by $\check{K} = \text{diag}(\check{k}_1, \check{k}_2, \check{k}_3)$, and then the unitary matrix \check{U} which satisfies $\check{H}^f = \check{U} \check{K} \check{U}^\dagger$. Since this is standard textbook quantum mechanics, we leave this as an exercise for the reader. For reference, the differences between the eigenvalues are found to be $\check{k}_3 - \check{k}_1 = 3.835 \times 10^{-22}$ erg and $\check{k}_3 - \check{k}_2 = 3.715 \times 10^{-22}$ erg. Note that since \check{H} is a function of V_* , the eigenvalues and unperturbed mixing matrix, \check{U} are also functions of V_* . The level scheme we end up with is shown in figure (1) with the three eigenstates of the unperturbed system denoted as $|m_1\rangle$, $|m_2\rangle$ and $|m_3\rangle$.

The two sinusoids in the matter Hamiltonian are the Fourier decomposed perturbation. The C matrices for the two Fourier modes are

$$C_1 = \frac{1}{2} V_* A_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (34)$$

$$C_2 = \frac{1}{2} V_* A_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (35)$$

giving diagonal elements of the F_a matrices of the form

$$f_{a;i} = \frac{1}{2} |\check{U}_{ei}|^2 A_a V_* \quad (36)$$

while all the elements in the Φ_a matrices are $\pi/2$. This has the consequence that for both Fourier modes

$$y_{a;ij} = z_{a;ij} = \frac{A_a V_* (|U_{ei}|^2 - |U_{ej}|^2)}{2 q_a} \quad (37)$$

and therefore all of the phases $\psi_{a;ij}$ are also equal to $\pi/2$. The elements of the G_a matrices are

$$G_{a;ij} = \frac{1}{2} \check{U}_{ei}^* \check{U}_{ej} A_a V_* \quad (38)$$

If we put all these pieces together we find the λ ’s are simply

$$\lambda_{a,m_a;ij} = J_{m_a}(z_{a;ij}) \quad (39)$$

while the κ ’s are of the form

$$\kappa_{a,m_a;ij} = i [G_{a;ji} J_{m_a+1}(z_{a;ij}) + G_{a;ij} J_{m_a-1}(z_{a;ij})]. \quad (40)$$

Let us now set the wavenumbers for the two modes so that $q_1 \approx \check{k}_3 - \check{k}_1$ and $q_2 \approx \check{k}_3 - \check{k}_2$ as shown in figure (1). The RWA integers we select for the 13 element are thus $\{n_{1,13}, n_{2,13}\} = \{+1, 0\}$ and for the 23 element we pick $\{n_{1,23}, n_{2,23}\} = \{0, +1\}$. The integer set for the 12 element must therefore be $\{n_{1,12}, n_{2,12}\} = \{+1, -1\}$ in order that $n_{a,12} + n_{a,23} = n_{a,13}$. The Hamiltonian $H^{(B)}$ is thus

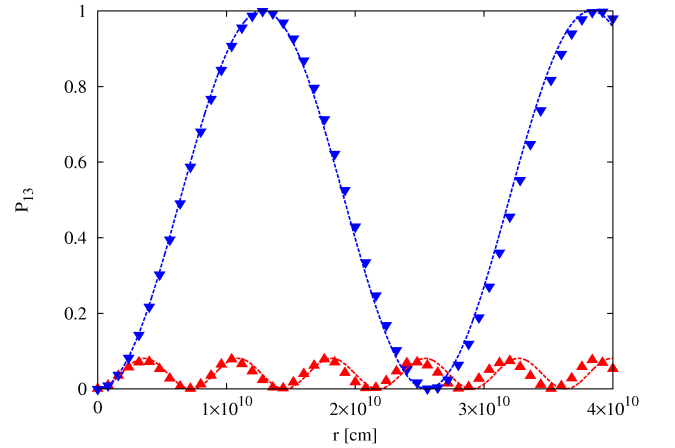


FIG. 2: The transition probabilities from mass state 1 to mass state 3. The blue line is for the case $A_1 = 0.1, A_2 = 0$ and the red for $A_1 = 0.1, A_2 = 0.5$.

$$H^{(B)} = \begin{pmatrix} 0 & -\imath\kappa_{12} e^{\imath(\check{k}_{12}+q_1-q_2)r} & -\imath\kappa_{13} e^{\imath(\check{k}_{13}+q_1)r} \\ \imath\kappa_{12}^* e^{-\imath(\check{k}_{12}+q_1-q_2)r} & 0 & -\imath\kappa_{23} e^{\imath(\check{k}_{23}+q_2)r} \\ \imath\kappa_{13}^* e^{-\imath(\check{k}_{13}+q_1)r} & \imath\kappa_{23}^* e^{-\imath(\check{k}_{23}+q_2)r} & 0 \end{pmatrix} \quad (41)$$

with

$$\begin{aligned} \kappa_{12} &= \kappa_{12,n_{1,12}} \lambda_{12,n_{2,12}} + \lambda_{12,n_{1,12}} \kappa_{12,n_{2,12}} \\ &= \imath [G_{1;21} J_2(z_{1;12}) + G_{1;12} J_0(z_{1;12})] J_{-1}(z_{2;12}) + \imath J_1(z_{1;12}) [G_{2;21} J_0(z_{2;12}) + G_{2;12} J_{-2}(z_{2;12})] \end{aligned} \quad (42)$$

$$\begin{aligned} \kappa_{13} &= \kappa_{13,n_{1,13}} \lambda_{13,n_{2,13}} + \lambda_{13,n_{1,13}} \kappa_{13,n_{2,13}} \\ &= \imath [G_{1;31} J_2(z_{1;13}) + G_{1;13} J_0(z_{1;13})] J_0(z_{2;13}) + \imath J_1(z_{1;13}) [G_{2;31} J_1(z_{2;13}) + G_{2;13} J_{-1}(z_{2;13})] \end{aligned} \quad (43)$$

$$\begin{aligned} \kappa_{23} &= \kappa_{23,n_{1,23}} \lambda_{23,n_{2,23}} + \lambda_{23,n_{1,23}} \kappa_{23,n_{2,23}} \\ &= \imath [G_{1;32} J_1(z_{1;23}) + G_{1;23} J_0(z_{1;23})] J_1(z_{2;23}) + \imath J_0(z_{1;23}) [G_{2;32} J_2(z_{2;23}) + G_{2;23} J_0(z_{2;23})] \end{aligned} \quad (44)$$

As reference, the numerical values we find are $\kappa_{12} = 6.419 \times 10^{-32} \imath$ erg, $\kappa_{13} = -3.888 \times 10^{-27} \imath$ erg and $\kappa_{23} = -1.311 \times 10^{-26} \imath$ erg. Once we recall that the Bessel function $J_n(z) \sim z^{|n|}$ for small z , we see that the element κ_{12} is smaller in magnitude than κ_{13} and κ_{23} . We now proceed to solve for B . Following the steps given above, the matrix M is

$$M = \begin{pmatrix} 0 & -\imath\kappa_{12} & -\imath\kappa_{13} \\ \imath\kappa_{12}^* & 0 & -\imath\kappa_{23} \\ \imath\kappa_{13}^* & \imath\kappa_{23}^* & 0 \end{pmatrix}. \quad (45)$$

Next we need the matrix Λ . In order to construct this matrix we need to find a set of six integers such that $n_{a,i} - n_{a,j} = n_{a,ij}$. One possible solution is $n_{1,1} = 1$, $n_{1,2} = 0$, $n_{1,3} = 0$ and $n_{2,1} = 0$, $n_{2,2} = 1$, $n_{2,3} = 0$. With this choice,

$$\Lambda = \begin{pmatrix} \check{k}_1 + q_1 & 0 & 0 \\ 0 & \check{k}_2 + q_2 & 0 \\ 0 & 0 & \check{k}_3 \end{pmatrix}, \quad (46)$$

Thus $H^{(\Omega)} = M + \Lambda$ is

$$H^{(\Omega)} = \begin{pmatrix} \check{k}_1 + q_1 & -\imath\kappa_{12} & -\imath\kappa_{13} \\ \imath\kappa_{12}^* & \check{k}_2 + q_2 & -\imath\kappa_{23} \\ \imath\kappa_{13}^* & \imath\kappa_{23}^* & \check{k}_3 \end{pmatrix}. \quad (47)$$

B has the formal solution $B(r) = \Upsilon(r) \exp(-\imath H^{(\Omega)} r) \Upsilon^\dagger(0)$ where $\Upsilon(r) = \exp(\imath \Lambda r)$ which can be evaluated using standard procedures. If we make the approximation that κ_{12} is negligibly small compared to κ_{13} and κ_{23} and that the two wavenumbers are exactly on resonance, $q_1 = \check{k}_3 - \check{k}_1$, $q_2 = \check{k}_3 - \check{k}_2$, then we find the analytical expression for the B matrix is

$$B = \exp\left(\imath \left[\Lambda - \check{k}_3 \mathbb{1}\right] r\right) \begin{pmatrix} \frac{|\kappa_{23}|^2}{Q^2} + \frac{|\kappa_{13}|^2}{Q^2} \cos(Qr) & \frac{\imath\kappa_{13}\kappa_{23}^*}{Q^2} [\cos(Qr) - 1] & -\frac{\kappa_{13}}{Q} \sin(Qr) \\ \frac{\imath\kappa_{23}\kappa_{13}^*}{Q^2} [\cos(Qr) - 1] & \frac{|\kappa_{13}|^2}{Q^2} + \frac{|\kappa_{23}|^2}{Q^2} \cos(Qr) & -\frac{\kappa_{23}}{Q} \sin(Qr) \\ \frac{\kappa_{13}^*}{Q} \sin(Qr) & \frac{\kappa_{23}^*}{Q} \sin(Qr) & \cos(Qr) \end{pmatrix}, \quad (48)$$

where $Q^2 = |\kappa_{13}|^2 + |\kappa_{23}|^2$ and $\mathbb{1}$ a 3x3 unit matrix. From this result we can extract the transition probability from mass state 1 to mass state 3 by taking the squared magnitude of B_{13}

$$P_{13} = \frac{|\kappa_{13}|^2}{Q^2} \sin^2(Qr) = \left(1 - \frac{|\kappa_{23}|^2}{Q^2}\right) \sin^2(Qr). \quad (49)$$

This result is interesting because it indicates the presence of the frequency q_2 which drives transitions between mass state 2 to mass state 3 has an influence on P_{13} . In the extreme case when κ_{23} is significantly larger than κ_{13} , the transition from 1 to 3 is essentially suppressed. This is an analog of the Electromagnetically Induced Transparency (EIT) - see, for example, [37–39] - in atomic physics where the presence of a second possible transition between atomic levels 2 and level 3 will inhibit the

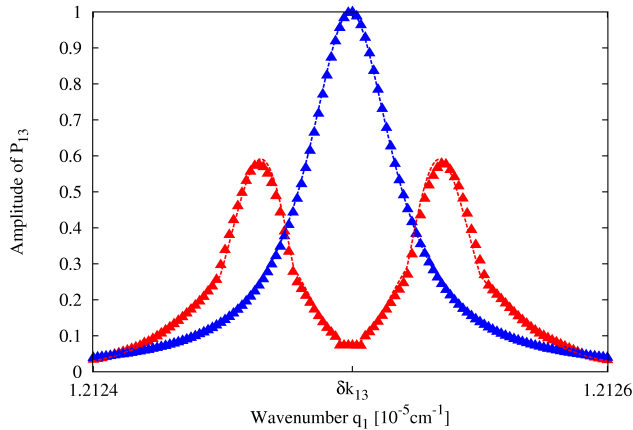


FIG. 3: The amplitude of P_{13} as a function of q_1 . The parameters used are $V_* = 6 \times 10^{-25}$ erg, $A_1 = 0.1$, and $A_2 = 0.5$ for the red dashed line and dots, $A_2 = 0$ for the blue dashed line and dots. The dots represent numerical results, while the dashed lines are from RWA evaluation.

primary transition from atomic level 1 to level 3 leading to little absorption, and thus transparency, for the light frequency corresponding to the energy splitting of level 1 and 3.

To illustrate this neutrino version of induced transparency, in Fig.(2) we plot the transition probability as a function of r when the system is at perfect resonance, namely when $q_1 = \check{k}_3 - \check{k}_1$ and $q_2 = \check{k}_3 - \check{k}_2$. The reader will observe that indeed, even though the frequency q_1 is on resonance with the transition between neutrino states 1 and 3, the probability of being in state 3 has a maximum of only 10%. If we remove the second sinusoid q_2 the transition probability P_{13} would increase to 100%. Note also a) that the solution is periodic even though the two frequencies q_1 and q_2 do not form rational ratio, and b) how well the numerical solution to the problem agrees with the RWA solution. The predicted amplitude and the frequency match the amplitude and frequency of the numerical solution to within a few percent.

To see the effect of induced transparency more clearly, we fix q_2 at the resonance between states 2 and 3 and scan in q_1 . The solution for B can be found by evaluating the formal solution and from the element B_{13} we extract the transition probability P_{13} . In Fig.(3) we plot the amplitude of the oscillations in P_{13} as a function of q_1 . We see that in the presence of mode q_2 , the transition probability has a peculiar shape with peaks off-resonance and local minimum at the resonance. If we turn off the second perturbing mode by setting A_2 to zero we recover the expected resonance at q_1 . Again, the RWA is able to reproduce the shape of P_{13} versus q_1 very well.

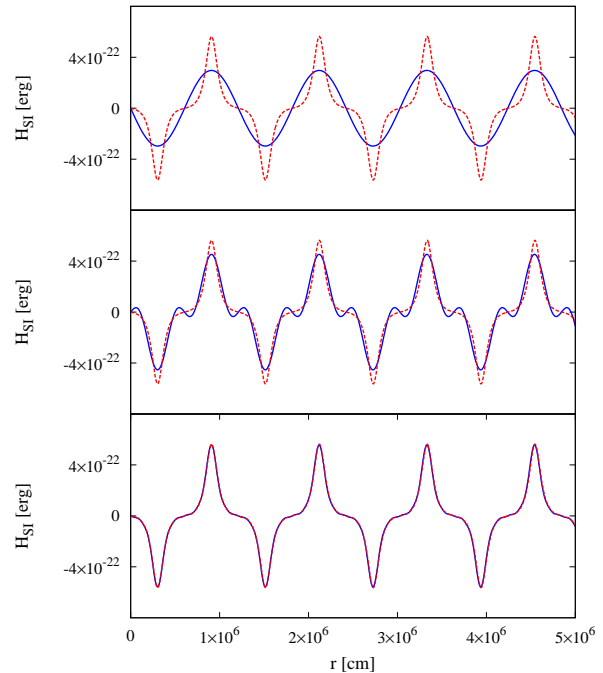


FIG. 4: The match between the Fourier reconstructed H_{SI} and the numerical H_{SI} . The top panel uses one Fourier mode, the middle panel uses two modes, and the bottom panel uses five modes. In all panels the dashed lines are the numerical result.

B. The Neutrino Self-Interaction Problem

The next problem we select is the case of neutrino flavor evolution including neutrino 'self-interaction'. To the vacuum term H_V and matter term H_M we add a third term coming from neutrino self-interactions H_{SI} [42]. Environments with sufficiently large neutrino densities include core-collapse supernovae and the early Universe. The effect of the first two contributions alone is well understood. The effect of neutrino self-interactions is much more complex because H_{SI} depends upon the evolution matrix one is trying to compute. This dependency makes the problem non-linear. In problems such as supernovae where there is a net neutrino current, one also has to include the effect that the weak interaction between two neutrinos depends upon the factor $1 - \cos \theta$ where θ is the angle between the neutrino trajectories. The numerical solutions for the flavor evolution including self-interaction have revealed an array of interesting behavior - see [40, 41] for reviews. Although no analytic solutions for the self-interaction problem in the early Universe nor supernovae are known, some important features of the system which determine its fate have been found. For example, the essential difference between neutrino evolution with self-interactions and neutrino evolution

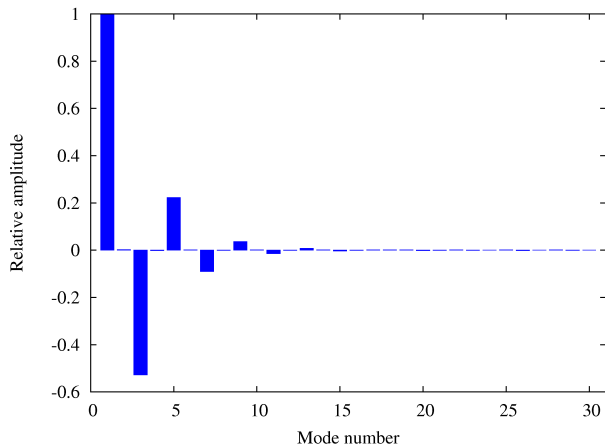


FIG. 5: The relative amplitudes of the Fourier components of the self-interaction potential.

through matter alone were pointed out in [43]. Also much recent effort has gone into a ‘stability analysis’ of the neutrino self-interaction problem which has revealed the conditions that lead to large neutrino flavor oscillations if not exactly what those oscillations look like [44–46].

It is not our intention here to try and apply the theory from section §II to the either the neutrino self-interaction problem in the early Universe nor supernovae. These problems require we deal with the many complications which would only obscure the issues related to the theory and so we reserve that application to a future study. Nevertheless, the neutrino self-interaction problem is an interesting one because, as we shall show, it requires we deal with issue of degeneracy in the RWA solution. To avoid the complicating factors of supernovae and early Universe neutrinos, we choose study the simplified self-interaction problem presented in Hannestad *et al.* [47] which considered a monoenergetic neutrino and antineutrino in vacuum only with a constant self-interaction strength μ . This is a well studied model that also forms the basis for the models used to study the matter-neutrino resonance [48–51]. The self-interaction Hamiltonian for the neutrinos in the flavor basis for this model is

$$\begin{aligned} H_{SI} &= \delta H = \mu [\rho(r) - \alpha \bar{\rho}^*(r)] \\ &= \mu [S\rho(0)S^\dagger - \alpha (\bar{S}\bar{\rho}(0)\bar{S}^\dagger)^*] \end{aligned} \quad (50)$$

where $\rho(r)$ and $\bar{\rho}(r)$ are the density matrices of the neutrinos and antineutrinos respectively at position r , S and \bar{S} the evolution matrices. For the antineutrinos the self-interaction Hamiltonian, \bar{H}_{SI} , is related to that of the neutrinos by $\bar{H}_{SI} = -H_{SI}^*$. The parameter α sets the asymmetry between the neutrinos and antineutrinos and we shall assume the density matrices at the initial point are 100% electron neutrino and electron antineutrino flavor. This is the model we shall adopt as a test for our methodology. For definiteness in what follows we set

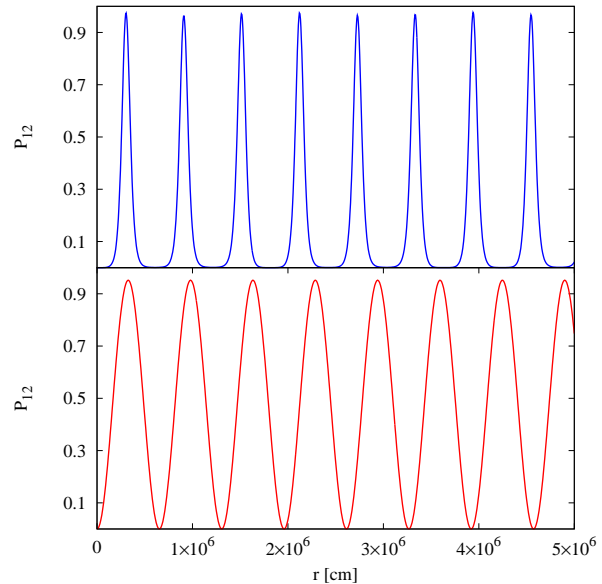


FIG. 6: The transition probabilities of electron neutrinos in the flavor basis in the symmetric case. Top panel, the numerical solution; bottom panel the RWA solution calculated using the five largest amplitude modes.

$\mu = 1.682 \times 10^{-21}$ erg, we use just two flavors. The vacuum Hamiltonian is our unperturbed Hamiltonian and for two neutrino flavors is simpler than the previous example and can be written as

$$H_V = \frac{m_1^2 - m_2^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} \quad (51)$$

The ‘mass splitting’ is taken to be $m_1^2 - m_2^2 = -2.43 \times 10^{-3}$ eV² and we adopt a vacuum mixing angle of $\theta = 1^\circ$. The energy of the neutrino and antineutrino are set to $E = 20$ MeV. The eigenvalues of this unperturbed Hamiltonian are $k_1 = (m_1^2 - m_2^2)/4E$ and $k_2 = -(m_1^2 - m_2^2)/4E$.

Our goal is to use our methodology to analyze this model. To do that we first solve for the evolution of the neutrinos and antineutrinos numerically recording the self-interaction Hamiltonian as we integrate. We then decompose the Hamiltonian into its Fourier components before inserting the wavenumbers and amplitudes into the RWA machinery in order to generate a solution which we can then compare with the numerical.

C. The symmetric case

Let us first look at the symmetric case, $\alpha = 1$, in which neutrinos and antineutrinos appear in equal quantities. The self-interaction potential we find numerically is shown by the dashed lines in Fig. (4). Only the imaginary parts of the off-diagonal entries of H_{SI} are

non-zero which has the consequence that the antineutrino self-interaction Hamiltonian is equal to the neutrino self-interaction Hamiltonian. Thus the antineutrinos evolve exactly the same way as the neutrinos in this particular case. The figure also shows the self-interaction Hamiltonian is clearly periodic though not sinusoidal. When we do the Fourier expansion to find the modes and their amplitudes we find the fundamental frequency is $q_1 = 3.28 \times 10^{-7} \text{ cm}^{-1}$ with an amplitude of $A_1 = 2.97 \times 10^{-22} \text{ erg}$ for the imaginary component of the off-diagonal elements. The relative amplitudes of the other modes are shown in Fig. (5) and it is interesting to note that only the odd modes are present in the spectrum.

Before we construct the RWA solution from these modes and amplitudes, we have to estimate how many Fourier modes should be kept. We can do this by directly trying to reproduce the potential with different numbers of modes and checking how close we are able to recover the Hamiltonian obtained numerically. Fig. (4) shows the comparison of the reproduced potentials with different number of modes. We see that for the symmetric case, $\alpha = 1$, just five modes are needed, i.e. the modes with 1, 3, 5, 7 and 9 times the fundamental wavenumber q_1 , are sufficient. We have verified this conclusion further by rewriting our self-interaction code to use the Fourier decomposed H_{SI} as an external potential and found the evolution matrix we obtain is the same as from the self-interaction code. Concretely, we substitute the original self-consistent potential with the following pseudo-potential

$$H_{SI} = \sum_{a=1}^5 (C_a e^{iq_a r} + C_a^\dagger e^{-iq_a r}) = \begin{pmatrix} 0 & -i \sum_{a=1}^5 A_a \sin(q_a r) \\ i \sum_{a=1}^5 A_a \sin(q_a r) & 0 \end{pmatrix}, \quad (52)$$

where A_a 's are the amplitudes of the largest five Fourier modes. Only sinusoidal term appear because of the symmetry of the potential. Thus the C_a matrices are

$$C_a = \frac{1}{2} A_a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (53)$$

The next step is to convert C_a matrices into G_a by using the matrices \check{U} which diagonalize the unperturbed Hamiltonian. In this test problem these matrices are known because we have set the matter Hamiltonian to zero so the \check{U} matrix is

$$\check{U} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (54)$$

With this matrix we find the G_a 's are also purely off-diagonal and equal to the C_a 's. Usually the next step in the RWA procedure would be to extract from the G_a 's the diagonal elements in order to compute the F_a 's and Φ_a 's

which are then inserted into Eqs. (11,12,13,14) to get x_a, y_a, z_a, ψ_a . However this is not necessary in this test problem because G_a matrices are all purely off-diagonal so we know immediately that $z_a = 0$ for all modes.

We now have to determine the set - or sets - of integers $\{n_1, n_2, n_3, n_4, n_5\}$ to use for the RWA equations. The approach we adopt is to scan the grid points of the five dimensional hypercube from $-2 \leq n_a \leq +2$ and select the combination that maximizes the amplitude of the RWA solution i.e. κ^2/Q^2 . We find this set to be $\{n_1, n_2, n_3, n_4, n_5\} = \{-1, 0, 0, 0, 0\}$. Normally the next step is to recognize that since the wavenumbers for the Fourier modes are all harmonics, their ratios are rational and thus we should be in a situation where we have degenerate RWA. There are other sets of integers which have the same detuning frequency as the set $\{-1, 0, 0, 0, 0\}$ and if we scan through the grid points of the hypercube defined by $-2 \leq n_a \leq +2$ we find there are sixty four other degenerate integer sets. However, as noted, the z_a 's for all modes are zero so the value of λ_{a,m_a} , equation (17), will also be zero except when $m_a = 0$. Since κ is a product of four λ_{a,m_a} 's, only *one* of the RWA integers can be non-zero; if two or more integers in the set are non-zero the contribution to the amplitude vanishes. Thus the set $\{-1, 0, 0, 0, 0\}$ is, in fact, the only

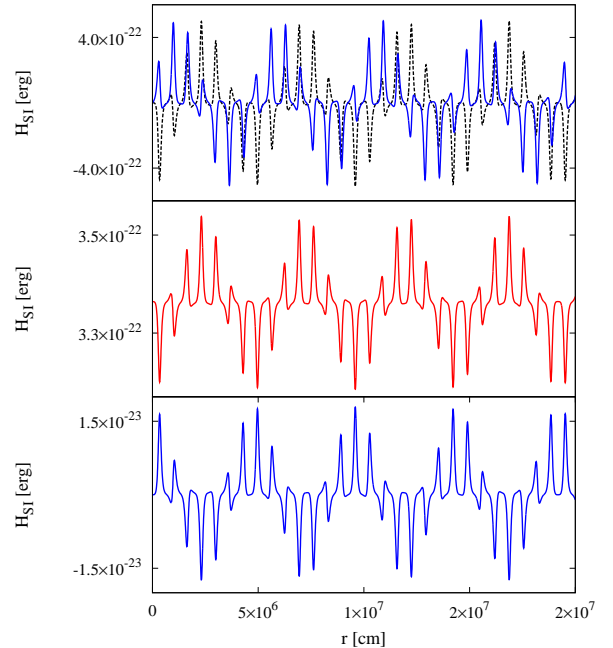


FIG. 7: The diagonal and off-diagonal entries of H_{SI} . Top panel: the imaginary (solid line) and real(dashed line) part of the off-diagonal entries. Middle panel: the real part of the first diagonal element. Bottom panel: the real parts of the second diagonal element.

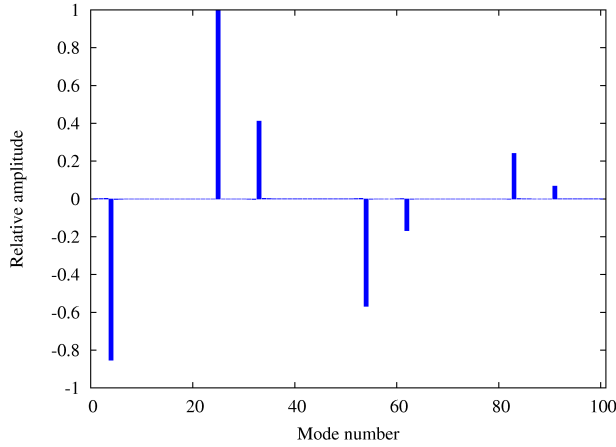


FIG. 8: The relative amplitudes of the Fourier components of the real part of the off-diagonal element of the Hamiltonian shown in Fig. 7.

set that contributes to the amplitude and it is not necessary to compute the contributions from the degenerate solutions.

Now that we know which set of RWA integers to use we simply compute the quantity κ and Q in order to determine the RWA solution. The comparison between the RWA solution and the transition probability as calculated numerically is shown in Fig. (6). Though the shape of the transition probability from the numerical solution and RWA do not match - the numerical solution varies approximately as $P_{12} \sim \sin^{12}(Q'r)$ whereas the solution from the RWA is that $P_{12} \propto \sin^2(Qr)$ - what we observe is that the amplitude of the two curves do match well and that the wavenumber Q matches to within $\sim 5\%$.

D. The asymmetric case

Let us now switch to a more demanding example of an asymmetric case with $\alpha = 0.8$. Our procedure is the same as in the symmetric case above but for $\alpha \neq 1$ we find the self-interaction Hamiltonian is now much more complicated with oscillatory behavior in every element of the matrix, as shown in Fig. (7). Nevertheless, the Hamiltonian is again seen to be periodic with a fundamental wavenumber of $q_1 = 5.19 \times 10^{-6} \text{ cm}^{-1}$. An analysis finds that all entries have essentially the same Fourier spectrum. The relative amplitudes of the modes for the real part of the off-diagonal element of the Hamiltonian are shown in Fig. (8). The amplitude of the modes have been normalized to the amplitude of the largest which has a wavenumber 25 times the fundamental. The seven modes with the largest amplitudes correspond to wavenumbers which are 4, 25, 33, 54, 62, 83, and 91 times the fundamental frequency. It is interesting to notice that the fundamental is not among these seven important modes and also to notice the pattern that the splitting between

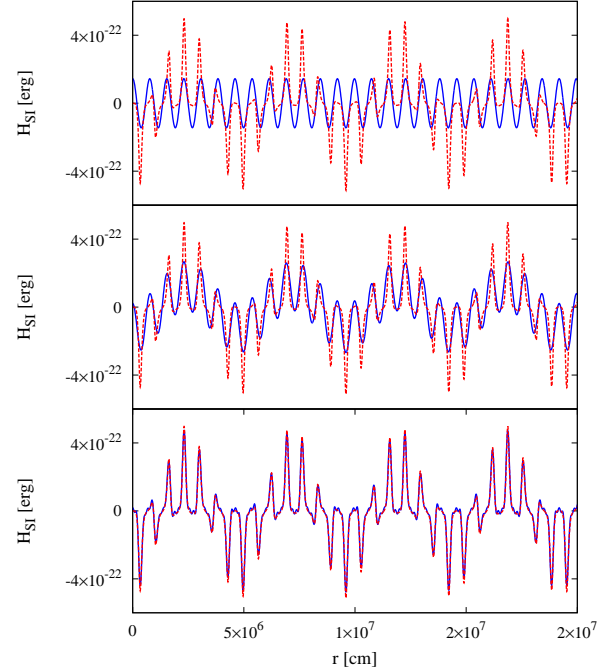


FIG. 9: The comparison between the real component of the off-diagonal element of the Fourier reconstructed H_{SI} and the numerical H_{SI} . In the top panel we use one Fourier mode, the middle panel is with two modes, and the bottom panel uses seven modes. In each panel the dashed line is the numerical result.

adjacent modes is 21,8,21,8,... times the fundamental wavenumber. As before we can determine the number of modes we need for the RWA by comparing the Fourier decomposed Hamiltonian with the numerical solution. The comparison is shown in Fig. (9) where we see we need the seven largest amplitude modes shown in Fig. (7) in order to reproduce the real part of the off-diagonal element of the Hamiltonian.

Now that we know the amplitudes and wavenumbers, we follow the same procedure as the symmetric case. This test problem is more demanding in several aspects. In this test problem the Hamiltonian for the antineutrinos is not equal to the neutrino Hamiltonian so we must compute the antineutrino evolution separately. Another difference is that the C_a and G_a matrices contain non-zero diagonal elements so the z_a 's are also non-zero. For the seven modes we are using we find z_a 's are all small, typically $z_a \sim 10^{-3}$. We scan through the $5^7 = 78,125$ grid points of the seven dimensional hypercube defined by $-2 \leq n_a \leq +2$ and find that the set integers which give the largest amplitude RWA solution are $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7\} = \{-1, 0, 0, 0, 0, 0, 0\}$. Unlike in the symmetric case of $\alpha = 1$, the contribution to the solution from the degenerate RWA integer

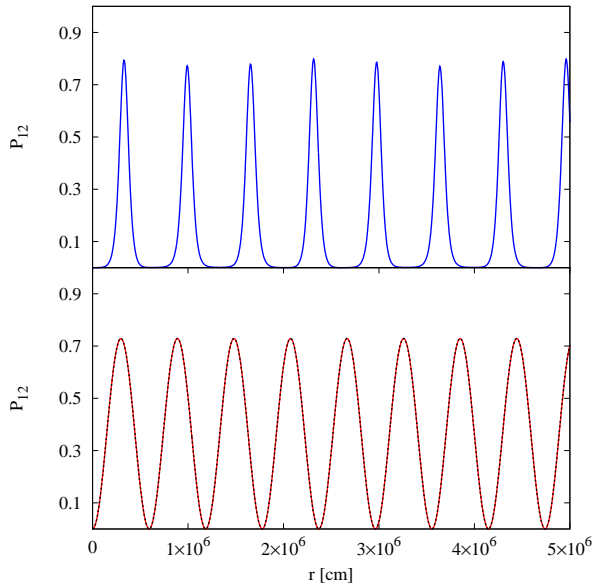


FIG. 10: The transition probabilities of electron neutrinos in the flavor basis in the asymmetric, $\alpha = 0.8$ case. Top panel, the numerical solution. Bottom panel the RWA solution using the seven largest amplitude modes. In the lower panel the solid line is the RWA solution using just the dominant RWA integer set, the dashed line is the solution computed with all the integer sets that have a degenerate detuning frequency.

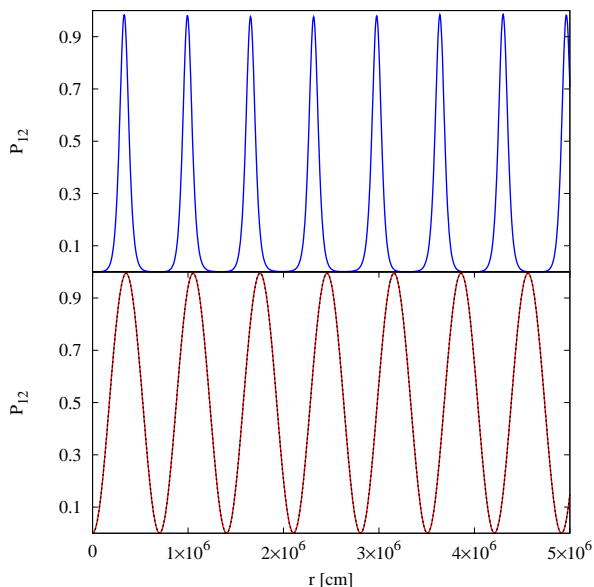


FIG. 11: The transition probabilities of electron antineutrino in the flavor basis for the same calculation as in figure (10).

sets can be non-zero. Within the seven dimensional hypercube we find 379 other sets with the same detuning frequency as the $\{-1, 0, 0, 0, 0, 0\}$ set ranging from the simple such as $\{1, 1, -1, 0, 0, 0\}$ to the exotic such as $\{2, -2, -1, 2, 1, -2\}$. We must add the contribution from these degenerate sets to our value for κ .

The numerical result and RWA prediction for the neutrinos are shown in Fig.10 and for the antineutrinos in Fig.11. Again the numerical solution is seen to be periodic function that resembles a ‘pinched’ sinusoid - a fit indicates the neutrino solution evolves as $P_{12} \sim \sin^{18}(Q' r)$ - which the RWA is not able to reproduce but, as before, the amplitude is calculated to within $\sim 10\%$ for the neutrinos, within $\sim 1\%$ for the antineutrinos and the wavenumber is accurate to $\sim 5\%$. It is interesting to note that the transition wavenumber for the neutrinos is overestimated i.e. too large, but underestimated for the antineutrinos. We also see that the amplitude for the oscillations in the antineutrinos is 100% i.e they are ‘on resonance’ but the neutrinos are not. Note this resonance is different from that described by Raffelt [52]. In figures (10) and (11) we show the RWA solution using just the dominant integer set and then the solution when we include all the degenerate solutions with the same detuning frequency. There is no discernible difference between the two indicating RWA degeneracy does not greatly modify the solution. This is due to the smallness of the z_a ’s: for small arguments the Bessel function J_n scales as $J_n \propto z^n$ so the λ_{a,n_a} ’s, equation (17), are suppressed for non-zero n_a .

IV. DISCUSSION AND CONCLUSION

Mathematical tools for calculating the effect of a perturbation upon a quantum system are a valuable resource for both theorist and experimentalist. In this paper we have presented one such tool that is able to take an arbitrary perturbing Hamiltonian that has been decomposed into Fourier modes and predict the evolution by using the Rotating Wave Approximation. We have placed no restriction on the dimensions of the perturbing Hamiltonian, the number of Fourier modes, the size and structure of the amplitudes nor the wavenumbers. These lack of restrictions make the method applicable to a wide range of problems including those where the Hamiltonian is aperiodic / turbulent (but Fourier-decomposable). We illustrated how our method works in practice by using it to analyze two simple transformation problems chosen to illustrate various practical aspects of our method and for their potential insight.

The first is the case of a three flavor neutrino passing through a matter profile composed of a constant and two sinusoidal fluctuations. We found the RWA was able to predict the amplitude and frequency of the transition probabilities between pairs of states to within a few percent. We also discovered the equivalent of induced transparency when the expected maximal oscillations between

a given pair of neutrino states could be switched off by the presence of a second resonant Fourier mode between another pair of states. The effect of induced transparency has obvious application to neutrinos and turbulence and is a area we shall pursue in future work.

The second test problem we studied is the neutrino-antineutrino self-interaction problem. In this problem the Hamiltonian is found to be composed of a harmonic Fourier series with just a few large amplitude modes. Again, the RWA is able to predict the amplitude of the transition probability and the transition wavelength to better than 10% in two test cases we considered. Further use of the RWA to analyze this problem will be presented

at a later time. Ultimately the RWA may be a foundation for a prediction of the transition wavenumber and amplitude of the transition probability oscillations given the initial conditions.

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